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Solutions of the inhomogeneous wave equation with unusual propagation character and global solution of the Poisson equation

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Abstract. Solutions of the inhomogeneous wave equation propagating into space-like wedge regions are used to construct solutions of the Poisson equation for arbitrary non-localized locally integrable inhomogeneities. This, for instance, allows for a general proof of existence of standard gauges for the classical electromagnetic field in four-dimensional spacetime.

1. Introduction

There may be occasions where you would like to know how to construct a global solution of the Poisson equation

$$\Delta\Phi(\vec{x}) = -4\pi\rho(\vec{x}) \tag{1}$$

on $\mathbb{R}^3$ for badly localized inhomogeneities ρ. As an example, consider the gauge problem in classical electrodynamics. Given a general field $F^{\mu\nu}(x)$ fulfilling the Maxwell equations

$$\partial_\mu F^{\mu\nu} = \jmath^\nu \qquad \partial^\mu \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} = 0$$

for some current $\jmath^\mu$, it is quite easy to represent this field on all of $\mathbb{R}^4$ in the form

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,. \tag{2}$$

A suitable potential $A^\mu = (A^0, \vec{A})$ is obtained by the well known *Poincaré construction*

$$A^\mu(x) = \int_0^1 s x_\nu F^{\nu\mu}(sx)\,\mathrm{d}s \tag{3}$$

no matter how rapidly the field might increase at infinity [4, p 10]. However, if you want to change from this Lorentz covariant gauge to the Coulomb gauge

$$\operatorname{div}\vec{A} = 0 \tag{4}$$

by a gauge transformation

$$A^\mu \longmapsto A'^\mu \stackrel{\text{def}}{=} A^\mu - \partial^\mu f$$

where f has to be a solution of the Poisson equation†

$$\Delta f_\tau = -4\pi\rho_\tau \tag{5}$$

† The Poisson equation is well defined for Schwartz distributions [8]. However, from the theory of pseudodifferential operators it is known that this general case can be solved if it can be solved for ρ_τ of C^∞-type [3, section 4]. So, for clarity, we consider only the latter case. Then also f_τ will be infinitely differentiable, since the Laplacian is an elliptic operator [1, section VI.C.3].

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depending smoothly on the time parameter τ, where

$$f_\tau(\vec{x}) \stackrel{\text{def}}{=} f(\tau,\vec{x}) \qquad \rho_\tau(\vec{x}) \stackrel{\text{def}}{=} \frac{1}{4\pi}\operatorname{div}\vec{A}(\tau,\vec{x})\,.$$

Here you cannot assume ρ_τ to be sufficiently localized for the very Coulomb solution

$$f_\tau(\vec{x}) = \int \frac{\rho_\tau(\vec{x}')}{\|\vec{x}-\vec{x}'\|}\,\mathrm{d}^3\vec{x}'$$

to exist. How, then, to construct a solution for ρ_τ with arbitrary increase at infinity? There is no general existence proof for the Coulomb gauge in standard textbooks such as [5]. There are lots of examples† allowing special constructions, but one would like to have a *general construction*.

The corresponding problem for the inhomogeneous wave equation

$$\Box\psi = \jmath \qquad \Box \stackrel{\text{def}}{=} \partial_0\partial_0 - \Delta \tag{6}$$

is much simpler. Here, an easy solution is the superposition $\psi = \psi^+ + \psi^-$ of the standard solutions

$$\psi^\pm(x) = \int D_{\text{ret}}(\pm x')\jmath^\pm(x-x')\,\mathrm{d}x' \qquad D_{\text{ret}}(x) \stackrel{\text{def}}{=} \frac{1}{2\pi}\theta(x^0)\delta(x^2) \tag{7}$$

of

$$\Box\psi^\pm = \jmath^\pm \qquad \jmath^\pm(x) \stackrel{\text{def}}{=} \theta(\pm x^0)\jmath(x)\,.$$

Existence of the integrals (7) is guaranteed by Huygens' principle [2]:

$$\operatorname{supp} D_{\text{ret}} \subset \left\{x \in \mathbb{R}^4 : x^0 = \|\vec{x}\|\right\} \tag{8}$$

(in four-dimensional spacetime). This suggests constructing solutions of the Poisson equation (1) of the form‡

$$\Phi(\vec{x}) = \int \psi(x^0+\tau,\vec{x})\,\mathrm{d}\tau \tag{9}$$

with suitable solutions ψ of (6) for

$$\jmath(x) = 4\pi h(x^0)\rho(\vec{x}) \qquad \int h(x^0)\,\mathrm{d}x^0 = 1\,.$$

These solutions must be sufficiently well behaved for $t \to \pm\infty$ at fixed $\vec{x}$ in order to allow for the reasoning

$$\Delta\Phi(\vec{x}) = -\Box\Phi(\vec{x}) = -\int \Box\Psi(x^0+\tau,\vec{x})\,\mathrm{d}\tau = -4\pi\int h(x^0+\tau)\rho(\vec{x})\,\mathrm{d}\tau = -4\pi\rho(\vec{x})\,.$$

For badly localized ρ, the constructions described in the standard literature such as [6, ch 5] do not tell us when this requirement is fulfilled. This is because $\Psi(x)$ picks up contributions from $\jmath(x')$ wherever $x^0 - x'^0 > 0 = (x-x')^2$. Even for fixed $\vec{x}$ (and compactly supported h) these contributions may become arbitrarily large for $x^0 \to \infty$. Therefore, the main topic of the present papers is to construct solutions of the inhomogeneous wave equation for which this effect is seen to be suppressed.

† In case $\rho(\vec{x}) = \rho(|\vec{x}|)$, for example, the problem becomes trivial in polar coordinates.

‡ For well localized ρ integrating $\psi = \psi^+ + \psi^-$ will give the Coulomb solution of (1), again.

2. Waves with unusual propagation character

In one space dimension there are no problems of the indicated type at all. Nevertheless, in order to get an idea, let us discuss this trivial case first.

Here, while (1) becomes completely trivial, the solution of the inhomogeneous wave equation

$$\left(\left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2\right) f(x^0, x^1) = \jmath(x^0, x^1) \tag{10}$$

may be reduced to elementary integration by use of the light-cone coordinates

$$\xi_\pm \stackrel{\text{def}}{=} x^0 \pm x^1 . \tag{11}$$

In these coordinates (10) becomes

$$4\frac{\partial}{\partial\xi_+}\frac{\partial}{\partial\xi_-}\hat{f} = \hat{\jmath} \tag{12}$$

where

$$\hat{f}(\xi_+, \xi_-) \stackrel{\text{def}}{=} f\left(\frac{\xi_+ + \xi_-}{2}, \frac{\xi_+ - \xi_-}{2}\right)$$

and similarly for $\hat{\jmath}$. Obviously, a special solution of (12) is given by

$$\hat{f}_\jmath(\xi_+, \xi_-) \stackrel{\text{def}}{=} \frac{1}{4}\int_0^{\xi_+}\left(\int_0^{\xi_-} \hat{\jmath}(\xi'_+, \xi'_-)\,\mathrm{d}\xi'_-\right)\mathrm{d}\xi'_+ .$$

This solution has the interesting stability property

$$\operatorname{supp}\hat{\jmath} \subset V_{\sigma\sigma'} \Longrightarrow \operatorname{supp}\hat{f}_\jmath \subset V_{\sigma\sigma'}$$

where

$$V_{\sigma\sigma'} \stackrel{\text{def}}{=} \left\{(\xi_+, \xi_-) \in \mathbb{R}^2 : \sigma\xi_+ \geqslant 0\, \sigma'\xi_- \geqslant 0\right\} \qquad \text{for} \quad \sigma, \sigma' \in \{+, -\} .$$

Rewritten in terms of x^0 and x^1, e.g. for $(\sigma, \sigma') = (+, -)$, this means

$$\operatorname{supp}\jmath \subset W^{(2)} \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^2 : \left|x^0\right| \leqslant x^1\right\} \implies \operatorname{supp} f_\jmath \subset W^{(2)} \tag{13}$$

for the special solution

$$f_\jmath(x^0, x^1) = \hat{f}_\jmath(x^0 + x^1, x^0 - x^1)$$

of (10). Therefore, given

$$\jmath(x^0, x^1) = -4\pi h(x^0)\rho(x^1)$$

with† $h \in \mathcal{D}(\mathbb{R})$ and locally integrable ρ, we may easily construct a solution $f = f_{\text{spec}}$ of (10), for which

$$\Phi(x^1) \stackrel{\text{def}}{=} \int f_{\text{spec}}(x^0, x^1)\,\mathrm{d}x^0$$

exists. To make this obvious, it is sufficient to check the case $x^1 < 1 \Longrightarrow \rho(\vec{x}) = 0$. Then, since h is compactly supported, the shift

$$\jmath_R(x^0, x^1) \stackrel{\text{def}}{=} \jmath(x^0, x^1 - R)$$

† Following Schwartz [8], we denote by $\mathcal{D}(\mathbb{R}^4)$ the space of C^∞ functions on $\mathbb{R}^4$ with compact support and by $\mathcal{S}(\mathbb{R}^4)$ the space of *tempered* functions, i.e. C^∞ functions on $\mathbb{R}^4$ with rapid decrease at infinity.

of j is supported by $W^{(2)}$ for sufficiently large R. This implies $\operatorname{supp} f_{jR} \subset W^{(2)}$ and therefore $\operatorname{supp} f_{\rm spec} \subset W^{(2)} - R$ for the special solution

$$f_{\rm spec}(x^0, x^1) \stackrel{\rm def}{=} f_{jR}(x^0, x^1 + R)$$

of (10). Obviously, then, the choice $\int h(x^0)\,\mathrm{d}x^0 = 1$ implies

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^1}\right)^2 \phi(x^1) = -4\pi\rho(x^1)\,. \tag{14}$$

Admittedly, this is a very crazy way of solving the trivial one-dimensional version (14) of (1). However, this method may be adjusted to work also in three space dimensions. Evidently, no additional problem arises if $j(x)$ is constant in x^2 and x^3, implying

$$\operatorname{supp} \tilde{j} \subset \left\{p \in \mathbb{R}^4 : \ p^2 = p^3 = 0\right\}$$

for its Fourier transform

$$\tilde{j}(p) \stackrel{\rm def}{=} (2\pi)^{-2} \int j(x) \mathrm{e}^{\mathrm{i}px}\,\mathrm{d}x\,.$$

This suggests the following generalization† of (13).

Lemma 2.1. Let $m \geqslant 0$ $R > 2m$, let $j \in \mathcal{S}(\mathbb{R}^4)$ fulfil the conditions

$$\operatorname{supp} j \subset W \stackrel{\rm def}{=} \left\{x \in \mathbb{R}^4 : \ |x^0| \leqslant x^1\right\} \qquad \operatorname{supp} \tilde{j} \subset \left\{p \in \mathbb{R}^4 : \ (p^2)^2 + (p^3)^2 < R^2/2\right\}$$

and define

$$e_1 \stackrel{\rm def}{=} (0, 1, 0, 0) \qquad \tilde{j}(p - \mathrm{i}Re_1) \stackrel{\rm def}{=} (2\pi)^{-2} \int j(x) \mathrm{e}^{\mathrm{i}(p - \mathrm{i}Re_1)x}\,\mathrm{d}x$$

$$px \stackrel{\rm def}{=} p^0 x^0 - \sum_{k=1}^{3} p^k x^k\,.$$

Then

$$f(x) \stackrel{\rm def}{=} -\mathrm{e}^{Rx^1} (2\pi)^{-2} \int \frac{\tilde{j}(p - \mathrm{i}Re_1)}{(p - \mathrm{i}Re_1)^2 - m^2} \mathrm{e}^{-\mathrm{i}px}\,\mathrm{d}p$$

is a C^∞ function fulfilling the conditions

$$\left(\Box + m^2\right) f = j \qquad \operatorname{supp} f \subset W\,. \tag{15}$$

Proof. The condition $\operatorname{supp} j \subset W$ guarantees that $\mathrm{e}^{-Rx^1} j(x)$ is a tempered function of x and, consequently, $\tilde{j}(p - \mathrm{i}Re_1)$ a tempered function of p. Since

$$\left|(p - \mathrm{i}Re_1 - \mathrm{i}q)^2 - m^2\right| R^2/4 - m^2 > 0 \qquad \text{for} \quad p \in \operatorname{supp} \tilde{j} \text{ and } q \in W \tag{16}$$

this shows, first of all, that $f(x)$ is C^∞ (even though not tempered) and, secondly, that

$$\left(\Box + m^2\right) f(x) = (2\pi)^{-2} \int \tilde{j}(p - \mathrm{i}Re_1) \mathrm{e}^{-\mathrm{i}(p - \mathrm{i}Re_1)x}\,\mathrm{d}p\,. \tag{17}$$

Now for fixed p^2, p^3 define

$$g_{p^2, p^3}(x^0, x^1) \stackrel{\rm def}{=} \frac{1}{2\pi} \int j(x) \mathrm{e}^{-\mathrm{i}(p^2 x^2 + p^3 x^3)}\,\mathrm{d}x^2\,\mathrm{d}x^3\,.$$

This is a tempered function of x^0, x^1 with

$$\operatorname{supp} g_{p^2, p^3} \subset \mathcal{W} \stackrel{\rm def}{=} \left\{(x^0, x^1) \in \mathbb{R}^2 : \ |x^0| \leqslant x^1\right\}.$$

† Thanks to the cut-off in (p^2, p^3) the (x^2, x^3)-independence of j, exploited above, is sufficiently well approximated.

Therefore its Fourier–Laplace transform

$$\tilde{j}(p^0+\mathrm{i}q^0, p^1+\mathrm{i}q^1, p^2, p^3) = \frac{1}{2\pi}\int g_{p^2,p^3}(x^0,x^1)\mathrm{e}^{+\mathrm{i}((p^0+\mathrm{i}q^0)x^0-(p^1+\mathrm{i}q^1)x^1)}\,\mathrm{d}x^0\mathrm{d}x^1$$

is an analytic function of $(p^0+\mathrm{i}q^0, p^1+\mathrm{i}q^1)$ in the open† tube $\mathbb{R}^2 - \mathrm{i}\,\underline{\mathcal{W}}$ and for every integer N there is a polynomial $P_{N,p^2,p^3}(q^0,q^1)$ for which

$$\sup_{p^0,p^1\in\mathbb{R}} \left(1+\|(p^0,p^1)\|\right)^N \left|\tilde{j}(p^0-\mathrm{i}q^0, p^1-\mathrm{i}q^1, p^2, p^3)\right| \leqslant P_N(q^0,q^1) \qquad \forall (q^0,q^1)\in\underline{\mathcal{W}}. \tag{18}$$

This allows deformation of the path for the $(p^1-\mathrm{i}q^1)$-integration in the complex plane to get

$$\frac{1}{2\pi}\int \tilde{j}(p-\mathrm{i}Re_1)\mathrm{e}^{-\mathrm{i}p^0x^0}\mathrm{e}^{+\mathrm{i}(p^1-\mathrm{i}R)x^1}\mathrm{d}p^0\,\mathrm{d}p^1 = \frac{1}{2\pi}\int \tilde{j}(p)\mathrm{e}^{-\mathrm{i}p^0x^0}\mathrm{e}^{+\mathrm{i}p^1x^1}\,\mathrm{d}p^0\,\mathrm{d}p^1$$
$$= g_{p^2,p^3}(x^0,x^1)\,. \tag{19}$$

Therefore

$$j(x) = \frac{1}{2\pi}\int g_{p^2,p^3}(x^0,x^1)\mathrm{e}^{+\mathrm{i}(p^2x^2+p^3x^3)}\,\mathrm{d}p^2\,\mathrm{d}p^3$$
$$= (2\pi)^{-2}\int \tilde{j}(p-\mathrm{i}Re_1)\mathrm{e}^{-\mathrm{i}(p-\mathrm{i}Re_1)x}\,\mathrm{d}p\,.$$

By equation (17) this implies

$$j(x) = \left(\Box + m^2\right)f(x)\,.$$

So we are left to prove $\operatorname{supp} f \subset W$. Consider any fixed $x\in\mathbb{R}^4\setminus W$. Then, for suitable $q\in\underline{\mathcal{W}}\times\{(0,0)\}$ we have $qx>0$. By straightforward generalization of (19) we get

$$f(x) = -(2\pi)^{-2}\int \frac{\tilde{j}(p-\mathrm{i}Re_1-\mathrm{i}\lambda q)}{(p-\mathrm{i}Re_1-\mathrm{i}\lambda q)^2-m^2}\mathrm{e}^{-\mathrm{i}(p-\mathrm{i}Re_1-\mathrm{i}\lambda q)x}\,\mathrm{d}p^0\,\mathrm{d}p^1$$

for all $\lambda>0$. Since, by (16) and (18), the right-hand side vanishes for $\lambda\to+\infty$ this proves $f(x)=0$ for such x. ∎

Allowing for $m>0$ in lemma 2.1—which would not be necessary for our purpose—emphasizes the strange propagation character of the solution:

> Usually, solutions of the Klein–Gordon equation describe the propagation of particles (or antiparticles) with rest mass $m>0$ and, therefore, rapidly decrease outside the light-cone [7, p 157] rather than inside the light-cone.

3. Solutions of the Poisson equation

As explained in section 1 we want to construct solutions of (6), which may be time-averaged, for inhomogeneities of the form

$$j(x) = -4\pi h(x^0)\rho(\vec{x}) \qquad h\in\mathcal{D}(\mathbb{R}^1) \quad \rho \text{ infinitely differentiable}. \tag{20}$$

Every such j can be written as a finite sum of C^∞-functions g_υ which have either compact support or support contained in the interior of some cone of the form

$$C_K \stackrel{\text{def}}{=} \{\lambda x : \lambda\geqslant 0\,,\ x\in K\} \tag{21}$$

where K, depending on g_υ, results by some spatial rotation around the origin from a compact subset of $\underline{W}$. Therefore, our problem is essentially solved by the following.

† As usual, the interior of a set M is denoted by $\underline{M}$.

Lemma 3.1. Let $m \geqslant 0$, and let $\jmath$ be a C^∞ function vanishing outside C_K for some compact subset K of $\underline{W}$. Then there are C^∞ functions f, φ on $\mathbb{R}^4$ fulfilling the conditions

$$\left(\Box + m^2\right) f = \jmath - \varphi \qquad \operatorname{supp} f \subset W \quad \varphi \in \mathcal{S}(\mathbb{R}^4).$$

Proof. Since

$$C_K \subset \bigcup_{\nu=0}^{\infty}(W + \nu e_1)$$

and since $C_K \cap \mathbb{R}^4 \setminus (W + \nu e_1)$ is bounded for all $\nu \in \mathbb{Z}_+$, we may represent $\jmath$ in the form

$$\jmath(x) = \sum_{\nu=0}^{\infty} \jmath_\nu(x) \qquad x \in \mathbb{R}^4$$

with

$$\jmath_\nu \in \mathcal{D}(\mathbb{R}^4) \qquad \operatorname{supp} \jmath_\nu \subset W + \nu e_1$$

for $\nu \in \mathbb{Z}_+$. Next we approximate the $\jmath_\nu$ by tempered functions χ_ν fulfilling the conditions

$$\operatorname{supp} \chi_\nu \subset W + \nu e_1$$

and

$$\operatorname{supp} \tilde{\chi}_\nu \subset \left\{p \in \mathbb{R}^4 : (p^2)^2 + (p^3)^2 < R_\nu^2/2\right\} \qquad \text{for} \quad R_\nu \text{ large enough}.$$

This approximation may be done sufficiently rapidly (w.r.t. $\nu \to \infty$) to ensure convergence of

$$\varphi \stackrel{\text{def}}{=} \sum_{\nu=0}^{\infty} (\jmath_\nu - \chi_\nu)$$

in the topology of $\mathcal{S}(\mathbb{R}^4)$. By lemma 2.1, then, we easily get C^∞ solutions f_ν of

$$\Box f_\nu = \chi_\nu$$

with

$$\operatorname{supp} f_\nu \subset W + \nu e_1 .$$

With these f_ν, obviously,

$$f(x) \stackrel{\text{def}}{=} \sum_{\nu=0}^{\infty} f_\nu(x) \quad \forall x \in \mathbb{R}^4$$

fulfills the required conditions. ∎

Corollary 3.2. Let $\jmath$ be a C^∞ function vanishing outside a time-slice region $\{x \in \mathbb{R}^4 : |x^0| \leqslant \tau\}$. Then there are C^∞ functions f, φ on $\mathbb{R}^4$ fulfilling the conditions

$$\Box f = \jmath - \varphi \qquad \operatorname{supp} f \subset \mathbb{R}^4 \setminus \underline{V} \quad \varphi \in \mathcal{S}(\mathbb{R}^4). \tag{22}$$

Proof. Choose any compact subset $K \subset \underline{W}$ with $e_1 \in \underline{K}$. Then, for sufficiently large n, there are spatial rotations $\Lambda_1, \ldots, \Lambda_n$ and C^∞ functions $\check{\jmath}_1, \ldots, \check{\jmath}_n$ fulfilling the conditions

$$\operatorname{supp} \check{\jmath}_\nu \subset C_K$$

and†

$$\jmath - \sum_{\nu=1}^{n} \Lambda_\nu \check{\jmath}_\nu \in \mathcal{D}(\mathbb{R}^4) . \tag{23}$$

Now, by lemma 3.1, for every $\nu \in \{1, \ldots, n\}$ there are C^∞ functions $\check{f}_\nu, \check{\varphi}_\nu$ fulfilling

$$\Box \check{f}_\nu = \check{\jmath}_\nu - \check{\varphi}_\nu \qquad \operatorname{supp} \check{f}_\nu \subset W \quad \check{\varphi}_\nu \in \mathcal{S}(\mathbb{R}^4) .$$

Exploiting rotational invariance of the wave operator $\Box$ we therefore have

$$\Box f_\nu = \jmath_\nu - \varphi_\nu \qquad \operatorname{supp} f_\nu \subset \mathbb{R}^4 \setminus \underline{V} \quad \varphi_\nu \in \mathcal{S}(\mathbb{R}^4)$$

where

$$f_\nu \stackrel{\text{def}}{=} \Lambda_\nu \check{f} \qquad \jmath_\nu \stackrel{\text{def}}{=} \Lambda_\nu \check{\jmath} \qquad \varphi_\nu \stackrel{\text{def}}{=} \Lambda_\nu \check{\varphi} .$$

By equation (23), this shows that

$$f \stackrel{\text{def}}{=} \sum_{\nu=1}^{n} f_\nu \qquad \varphi \stackrel{\text{def}}{=} \jmath - \sum_{\nu=1}^{n} \jmath_\nu + \sum_{\nu=1}^{n} \varphi_\nu$$

fulfil the required conditions. ■

Now it is very easy to prove our main result:

Theorem 3.3. For every C^∞ function ρ on $\mathbb{R}^3$ there is a C^∞ solution Φ of the Poisson equation (1).

Proof. Choose any $h \in \mathcal{D}(\mathbb{R}^1)$ with

$$\int h(x^0)\,\mathrm{d}x^0 = 1$$

and define $\jmath$ by (20). By corollary 3 there are C^∞ functions f, φ fulfilling (22). By equations (6) and (7), we therefore have

$$f_\jmath(x) \stackrel{\text{def}}{=} f(x) - \int D_{\text{ret}}(x')\varphi(x - x')\,\mathrm{d}x' \qquad \forall x \in \mathbb{R}^4$$

is a solution of

$$\Box f_\jmath(x) = -4\pi h(x^0)\rho(\vec{x})$$

which may be integrated over time to give a solution

$$\Phi(\vec{x}) \stackrel{\text{def}}{=} \int f_\jmath(x^0, \vec{x})\,\mathrm{d}x^0$$

of equation (1). ■

Generalization to arbitrary locally integrable inhomogeneities ρ is straightforward.

† As usual, we define

$$\Lambda f(x) \stackrel{\text{def}}{=} f(\Lambda^{-1}x) \quad \forall x \in \mathbb{R}^4$$

for functions on $\mathbb{R}^4$ and Lorentz transformations Λ.

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